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THE VALUE OF INFORMATION GIVEN DECISION FLEXIBILITY

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THE VALUE OF INFORMATION GIVEN DECISION FLEXIBILITY

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper presents a useful concept for decision analysis -- the value of information given flexibility.

An exploration is made into the impact of decision flexibility on the value of information. The usefulness of calculating the value of information under various assumptions concerning decision flexibility is illustrated with a simple economic example. An upper limit to the value of information given flexibility is the expected value of perfect information given perfect

Block 20 (continued)

flexibility (EVPIGPF). By approximating an arbitrary smooth value function with a quadratic equation, first order characteristics of the EVPIGPF are identified. Finally, it is shown that the technique of proximal decision analysis may be expanded to provide a simplified estimation of the EVPIGPF for large-scale decision problems.

Abstract

This paper presents a useful concept for decision analysis -- the value of information given flexibility.

An exploration is made into the impact of decision flexibility on the value of information. The usefulness of calculating the value of information under various assumptions concerning decision flexibility is illustrated with a simple economic example. An upper limit to the value of information given flexibility is the expected value of perfect information given perfect flexibility (EVPIGPF). By approximating an arbitrary smooth value function with a quadratic equation, first order characteristics of the EVPIGPF are identified. Finally, it is shown that the technique of proximal decision analysis may be expanded to provide a simplified estimation of the EVPIGPF for large-scale decision problems.

1. Introduction

The well-known "value of information" concept of decision analysis provides a logical technique for placing a dollar value on the resolution of uncertainty. Normally this value is considered to be a constant against which the cost of obtaining information is compared. More generally, the value one places on information will depend upon one's assumed flexibility to make use of the information. The more flexible one's decisions are, the more valuable is information.

This paper presents a definition of decision flexibility for the science of decision analysis. A simple economic example is used to demonstrate the usefulness of calculating the value of information under various assumptions concerning decision flexibility. Howard [3] has suggested "proximal decision analysis" as a technique for analyzing large-scale decision problems when states and decisions can be represented by continuous vectors. The proximal model is used to analyze the effect of various problem characteristics on the value of information given flexibility.

2. A Definition of Decision Flexibility

The concept of flexibility has occasionally cropped up in micro-economic literature on the theory of the firm. For several approaches see Ref. [1, 4, 7, 8, 9]. For the purposes of decision analysis it is convenient to take a different approach. We view the flexibility of a given decision variable to be determined by the size of the choice set associated with that variable.

Let D and D' be two possible sets of feasible alternatives for a decision d and suppose that D is a proper subset of D'. Then the decision d is said to be more flexible in the case of the feasible set D' than in the case of the feasible set D. Roughly speaking, the larger the choice set -- that is, the more alternatives that are available for a decision -- the greater is the decision flexibility.

3. Value of Flexibility

Notation and Basic Decision Model

When dealing with the uncertainty in a decision, it is frequently important to state explicitly the information upon which a probability assessment is based. Inferential notation is useful for this purpose. Following [2], if x is a random variable, the symbol $\{x|S\}$ denotes the probability density function of x given the state of knowledge S. The expectation of x based on S is written $\langle x|S\rangle$. A special state of knowledge is the total prior experience available at the beginning of the problem. The total prior experience is denoted by $\mathcal{E}$.

We envision a decision model of the form shown in Figure 3.1 and discussed in Ref. [2]. Problem variables have been divided into those under the control of the decision maker -- decision variables $d_1,..,d_m$ -- and those not under his control - state variables $s_1,..,s_n$. The function $v(\underline{s},\underline{d})$ represents the decision maker's value model. For specified values of $\underline{s}$ and $\underline{d}$ it assigns a scalar value v. State variables are uncertain and described by a distribution $\{\underline{s}|\mathcal{E}\}$. For any given decision vector $\underline{d}$ a profit lottery $\{v|\underline{d},\mathcal{E}\}$ is produced on outcome value. The decision maker's preferences interact with this lottery so as to produce a utility measure $u(\{v|\underline{d},\mathcal{E}\})$. The objective for the decision maker is to choose from the feasible decision set $\underline{D}$ the decision vector $\underline{d}$ which produces the highest utility measure.

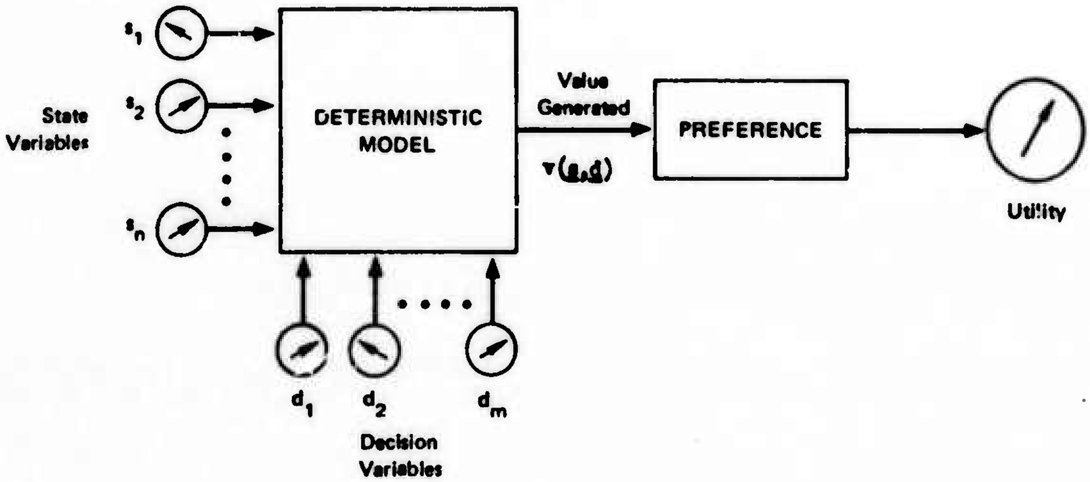

FIGURE 3.1 AN ABSTRACT REPRESENTATION OF THE BASIC DECISION MODEL

The Value of Information Given Flexibility

The value of information given flexibility measures the value to the decision maker, in economic units, of obtaining a given amount of information together with a given amount of decision flexibility. An upper limit to this quantity, the expected value of perfect information given perfect flexibility (EVPIGPF), may be calculated.

As a base case consider the decision problem in which the decision maker must set $\underline{d}$ prior to learning the state variable outcomes $\underline{s}$. Now consider the problem in which the decision maker may delay the setting of the j'th decision variable until after he learns the outcome of the i'th state variable. All other decisions, however, must be set prior to learning any state variable outcomes. We define the value of perfect information on s_i given perfect flexibility on d_j as the maximum number of economic units the decision maker would be willing to pay to change the structure of his decision from that of the first problem considered to that of the second. The flexibility is said to be perfect because it is assumed that receipt of the information does not restrict in any way the feasible decision set associated with the flexible decision variable.

The EVPIGPF is similar to, but more complete than, the concept of expected value of perfect information (EVPI). Whereas EVPI measures the value of perfect information under the assumption that all decision variables may be adjusted to utilize the information, the EVPIGPF explicitly states which decision variables may be adjusted in response to what information. In a real system it may be costly or impossible to maintain flexibility on all decisions while awaiting the arrival of some piece of information. By comparing the costs of maintaining flexibility with the EVPIGPF, the decision maker has a method for deciding which decisions ought to be kept flexible and for which it is more profitable to eliminate flexibility. We illustrate this use with a simple economic example.

Example: The Entrepreneur's Price-Quantity Decision

An entrepreneur must decide upon a price and quantity for his product. He is uncertain about the total cost c per item but feels that it may be represented by the uniform distribution of Fig. 3.2. He knows that the demand for his product will be a decreasing function of his price, but for any given price he is uncertain as to the exact quantity of his product demanded. For this reason he hypothesizes the following functional form for demand x :

$$x = \frac{a}{p} - b - e \ , \tag{3.1}$$

where

x = demand (in thousands of units),
p = price (in thousands of dollars),
a,b = parameters of the demand curve, and
e = a random variable independent of c and uniformly distributed from zero to one.

Figure 3.3 shows the probability density for e and the demand curve $x(p)$.

Further let

q = quantity produced (in thousands of units)
v = net profit (in millions of dollars) .

Then,

$$v(p,q,c,e) = \begin{cases} p(\frac{a}{p} - b - e) - cq \ , & \text{if } \frac{a}{p}-b-e < q \\ (p - c)\, q \ , & \text{if } \frac{a}{p}-b-e > q \end{cases} \ . \tag{3.2}$$

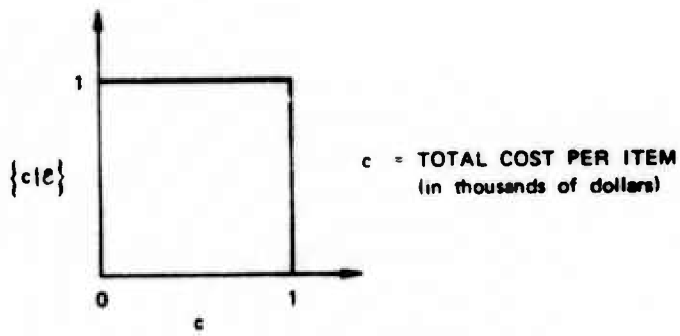

FIGURE 3.2 PROBABILITY DENSITY FUNCTION FOR PRODUCTION COST IN THE ENTREPRENEUR'S DECISION

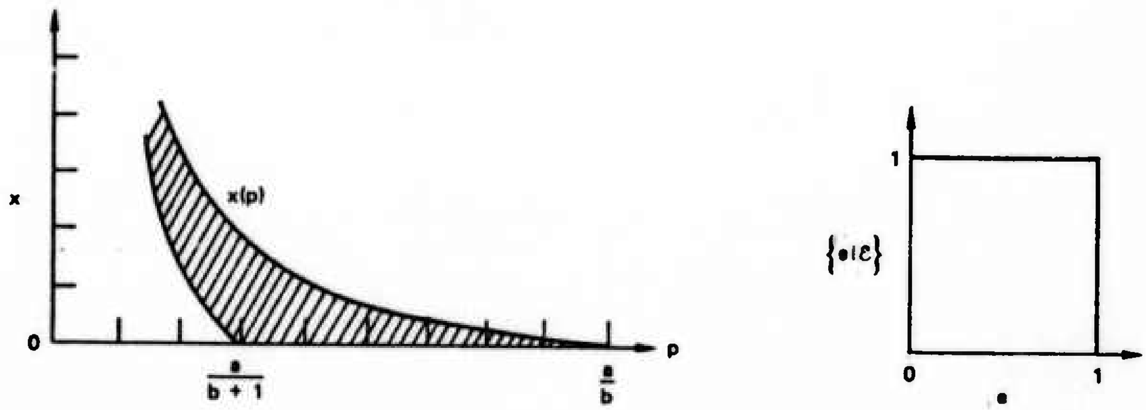

FIGURE 3.3 THE DEMAND CURVE AND THE PROBABILITY DENSITY FUNCTION FOR THE DEMAND PARAMETER e IN THE ENTREPRENEUR'S DECISION

We wish to determine our entrepreneur's expected net profit and the value to him of using various perfect information - perfect flexibility structures. In other words, we would like to know how much it is worth to the entrepreneur to obtain perfect information on various state variables if he uses that information when setting various decision variables. For example, consider the value to the decision maker of obtaining perfect information on the demand parameter e for the purpose of setting his production quantity q. This value is obtained by calculating the increment to expected profit produced by clairvoyance on e given flexibility on q:

$$\langle v|CeFq,\mathcal{E}\rangle - \langle v|\mathcal{E}\rangle = \max_{p} \int_{e} \max_{q} \int_{c} v \{c|\mathcal{E}\} \{e|\mathcal{E}\}$$

$$- \max_{p} \max_{q} \int_{c} \int_{e} v \{e|\mathcal{E}\} \{c|\mathcal{E}\} \qquad (3.3)$$

In all there are 3x3=9 possible perfect information-flexibility structures. The computations have been performed and are summarized in Table 3.1 for particular parameter values of $a=2.25$ and $b=0.5$.

Observe that the expected value of the entrepreneurial venture is half a million dollars and is obtained through an optimal decision strategy of setting price at $1,000 and quantity at 1,250 units. The entries in the Table -- V_{CF}, p^*, and q^* -- respectively denote the value of the information-flexibility structure and the optimal decision strategy appropriate to the structure corresponding to a given location in the Table. For example, if it were possible for our entrepreneur to pick a price, learn the demand parameter e, and then set his quantity, he could expect to increase his profits by $128,680. To do this he would set price at 1.061 thousand dollars, conduct his estimation of e, and then set quantity at $1.621 - e$ thousands of units.

The entries V_{CF} thus indicate the value to the decision maker of applying additional information to the various decisions that make up his problem. We observe that the greatest increase in profit

TABLE 3.1.

Value of and Decision Strategy for Various Information-Flexibility Structures in the Entrepreneur's Decision

Information About \ Flexibility On	p	q	p and q
c	$V_{CF} = \$0$ $p^* = 1.0$ $q^* = 1.25$	$V_{CF} = \$42,411$ $p^* = .964$ $q^* = \begin{cases} 0 & \text{if } c > .964 \\ 1.833-1.037c & \text{otherwise} \end{cases}$	$V_{CF} = \$139,416$ $p^* = \sqrt{\frac{(.45-c)c}{2}}$ $q^* = \frac{2.25-c}{\sqrt{\frac{(.45-c)c}{2}}} - .5$
e	$V_{CF} = \$151,639$ $p^* = \frac{2.25}{1.595+e}$ $q^* = 1.095$	$V_{CF} = \$128,680$ $p^* = 1.061$ $q^* = 1.621-e$	$V_{CF} = \$151,924$ $p^* = \sqrt{\frac{2.25}{1+2e}}$ $q^* = \sqrt{.45(.5+e)} - .5-e$
c and e	$V_{CF} = \$151,639$ $p^* = \frac{2.25}{1.595+e}$ $q^* = 1.095$	$V_{CF} = \$128,680$ $p^* = 1.061$ $q^* = 1.621-e$	$V_{CF} = \$271,915$ $p^* = \sqrt{\frac{2.25c}{.5+e}}$ $q^* = \sqrt{\frac{2.25(.5+e)}{c}} - .5-e$

$\langle v \mid \mathscr{E} \rangle = .5 = \$500,000$

$p^* = 1 = \$1,000$

$q^* = 1.25 = 1,250$ units

is expected if perfect information is obtained on both uncertain variables and that information is used to adjust both decision variables. Of the two pieces of information c and e, e is more valuable regardless of the flexibility assumed on the decision variables. The relative value of flexibility on the two decisions, however, depends on the information to be received. Flexibility on quantity is more valuable if the entrepreneur expects to learn costs. Flexibility on price is more valuable if the entrepreneur expects to learn demand or if he expects to learn both costs and demand.

Typically, a large number of information gathering and flexibility preserving schemes are available to the decision maker. Normally such schemes will provide imperfect rather than perfect information and less than complete flexibility. EVPIGPF's provide an upper bound to the value of such schemes and, therefore, allow the decision maker to dismiss immediately those whose costs exceed these bounds. Suppose that after considering various information gathering and decision delaying schemes, the entrepreneur constructs a table of proposals and costs as illustrated in Table 3.2. Proposals 1 and 2 can be eliminated from further consideration as their costs exceed the corresponding EVPIGPF's. Schemes 4 & 6 appear to be of dubious value while schemes 3, 7 and 9 are among those that appear to deserve further consideration.

Knowledge of EVPIGPF's can generate insight that is not provided by EVPI's alone. For example, observe from Table 3.1 that the value of clairvoyance on costs given flexibility on price is zero, but the value of clairvoyance on costs given flexibility on price and quantity is \$139,416. Information about costs is useful for setting price but only if that information is used for setting quantity as well. Once quantity has been fixed price must be set so as to clear the inventory, and costs are no longer a consideration. Insight may also be provided on decision timing. If information on the demand parameter e is purchased, virtually all the usefulness of the information, \$151,639 worth, can be obtained using it only to set price. Delaying production until after this information becomes available will only be worth an additional \$285 !

TABLE 3.2

Proposed Information Gathering-Flexibility Preserving Schemes for the Entrepreneur's Decision and Their Estimated Costs

Flexibility On / Information About	p	q	p and q
c	1 Delay advance publicizing of price until good estimate of costs is available. Cost = $5-40,000 from lost sales	2 Conduct prototype run to estimate costs before committing to full production. Cost = $45-60,000 in nonrecoverable costs	3 Conduct prototype run to estimate costs and delay advance publicizing of price. Cost = $60-90,000
e	4 Commit to quantity, conduct analysis of market potential of variously priced units. Cost = $100,000	5 Conduct market study to estimate demand at fixed price. Cost = $60-90,000	6 Conduct analysis of market potential and delay advance publicizing of price. Cost = $100,000
c and e	7 Conduct trial marketing test of variously priced units. Cost = $50-70,000	8 Conduct market study at fixed price and prototype run before committing to full production. Cost = $120-150,000	9 Conduct analysis of market potential and prototype run. Cost = $160,000

4. The Quadratic Decision Problem

In this section we shall obtain an explicit expression for the value of perfect information given perfect flexibility for a quadratic decision problem. A quadratic decision problem is defined as the basic decision model of Fig. 3.1 with the following additional assumptions:

1. The decision variables d_j are unconstrained.
2. The decision maker's value function $v(\underline{s},\underline{d})$ is a quadratic function in the s_i and d_j such that for every $\underline{s}$, $v(\underline{s},\underline{d})$ has a unique maximum with respect to $\underline{d}$.
3. The decision maker's utility function is a linear function of value.

The results will be shown to have a practical use in § 5.

By assumption (2),

$$v(\underline{s},\underline{d}) = a + \underline{b}'\underline{s} + \frac{1}{2}\underline{s}'W\underline{s} + \underline{s}'T\underline{d} + \underline{r}'\underline{d} + \frac{1}{2}\underline{d}'Q\underline{d} , \qquad (4.1)$$

with Q negative definite. Since we are interested in relative values, we may ignore the first three terms in (4.1). Further, by assuming that decision settings are measured as deviations from the best deterministic decision and state variables are measured from their mean values, there is no loss in generality if we take

$$v(\underline{s},\underline{d}) = \underline{s}'T\underline{d} + \frac{1}{2}\underline{d}'Q\underline{d} \qquad (4.2)$$

with $E(s) = \underline{0}$.

To characterize the various structures we use the following notation. Let $N = \{1,\ldots,n\}$ and $M = \{1,\ldots,m\}$ be the respective sets of state and decision variable indices. Define $I \subset N$ to be the set of indices of those state variables upon which information is to be obtained, and let $J \subset M$ denote the indices of decision variables for which flexibility is available. $\bar{I}$ and $\bar{J}$ will denote the complements within N and M of the sets I and J respectively. $Cs_I Fd_J$ will denote the information structure within which the decision maker has clairvoyance on state variables s_i, $i \in I$ and flexibility on decision variables d_j, $j \in J$.

For a given structure $Cs_I Fd_J$, let T_{IJ} denote the matrix $[t_{ij}]_{i \in I, j \in J}$ of those elements t_{ij} of T such that i is in I and j is in J, and similarly define T_{NJ}, Q_{JJ}, $Q_{\bar{J}\bar{J}}$, etc. T'_{NJ} and Q^{-1}_{JJ} will be taken to mean the transpose of T_{NJ} and the inverse of Q_{JJ}, respectively. Also, let $\underline{s}_I$ denote the vector of those components s_i of $\underline{s}$ such that $i \in I$, and similarly define $\underline{d}_J$ and $\underline{d}_{\bar{J}}$. Then, subject to the various assumptions made above, we have the

<u>THEOREM</u>: For any information-flexibility structure $Cs_I Fd_J$, the optimal decision strategy $\underline{d}^*$ is given by

$$\underline{d}^*_{\bar{J}} = -(Q_{\bar{J}\bar{J}} - Q_{\bar{J}J}Q^{-1}_{JJ}Q_{J\bar{J}})^{-1}(T'_{\bar{I}\bar{J}} - Q_{\bar{J}J}Q^{-1}_{JJ}T'_{\bar{I}J})E(\underline{x}_{\bar{I}}) \,, \tag{4.3}$$

$$\begin{aligned}\underline{d}^*_J = {} & -(Q_{JJ} - Q_{J\bar{J}}Q^{-1}_{\bar{J}\bar{J}}Q_{\bar{J}J})^{-1}(T'_{\bar{I}J} - Q_{J\bar{J}}Q^{-1}_{\bar{J}\bar{J}}T'_{\bar{I}\bar{J}})E(\underline{x}_{\bar{I}}) \\ & -Q^{-1}_{JJ}[T'_{IJ}\underline{s}_I + T'_{\bar{I}J}[\underline{x}_{\bar{I}} - E(\underline{x}_{\bar{I}})]] \,,\end{aligned} \tag{4.4}$$

and the corresponding expected value of the structure is

$$\begin{aligned}\langle v_{Cs_I Fd_J} | \mathcal{E} \rangle = -\frac{1}{2}\,\mathrm{trace}\Big\{ & T_{NJ}Q^{-1}_{JJ}T'_{NJ}E(\underline{x}\underline{x}') \\ & + (T_{\bar{I}\bar{J}} - T_{\bar{I}J}Q^{-1}_{JJ}Q_{J\bar{J}})(Q_{\bar{J}\bar{J}} - Q_{\bar{J}J}Q^{-1}_{JJ}Q_{J\bar{J}})^{-1}(T'_{\bar{I}\bar{J}} - Q_{\bar{J}J}Q^{-1}_{JJ}T'_{\bar{I}J})E(\underline{x}_{\bar{I}})E(\underline{x}_{\bar{I}})\Big\} \,,\end{aligned} \tag{4.5}$$

where $\underline{x} = E(\underline{s}\,|\,\underline{s}_I)$.

PROOF: Equations (4.3), (4.4), and (4.5) follow from the evaluation of

$$<v|Cs_I Fd_J, \mathcal{E}> - <v|\mathcal{E}> \tag{4.6}$$

where

$$<v|\mathcal{E}> = \max_{\underline{d}} E(v) = \max_{\underline{d}} \left(\frac{1}{2}\,\underline{d}'Q\,\underline{d}\right) = 0 \ , \tag{4.7}$$

because Q is negative definite, and

$$<v|Cs_I Fd_J, \mathcal{E}> = \max_{\underline{d}_{\bar{J}}} E[\max_{\underline{d}_J} E(v|\underline{s}_I)]$$

$$= \max_{\underline{d}_{\bar{J}}} E\left\{\max_{\underline{d}_J}[\underline{x}'T_{N\bar{J}}\underline{d}_{\bar{J}} + \underline{x}'T_{NJ}\underline{d}_J + \frac{1}{2}\underline{d}_J'Q_{JJ}\underline{d}_J + \underline{d}_J'Q_{J\bar{J}}\underline{d}_{\bar{J}} + \frac{1}{2}\underline{d}_{\bar{J}}'Q_{\bar{J}\bar{J}}\underline{d}_{\bar{J}}]\right\} \ . \tag{4.8}$$

A detailed derivation is contained in Ref. [5].

COROLLARY 1: Under the structure $Cs_I Fd_J$

$$\underline{d}_{\bar{J}}^* = \underline{0} \ , \tag{4.9}$$

$$\underline{d}_J^* = -Q_{JJ}^{-1}T_{NJ}'\underline{x} \ , \tag{4.10}$$

$$<v_{Cs_I Fd_J}|\mathcal{E}> = -\frac{1}{2}\text{trace}\left\{T_{NJ}Q_{JJ}^{-1}T_{NJ}'E(\underline{xx}')\right\} \tag{4.11}$$

if any of the following conditions hold:

(a) $E(\underline{x}_{\bar{I}}) = \underline{0}$

(b) $\bar{J} = \emptyset$ (Complete flexibility)

(c) $\bar{I} = \emptyset$ (Complete information) .

COROLLARY 2: Under the structure $Cs_I Fd_J$

$$\underline{d}_J^* = \underline{0} , \tag{4.12}$$

$$\underline{u}_J^* = -Q_{JJ}^{-1} T_{IJ}' \underline{s}_I , \tag{4.13}$$

$$\langle v_{Cs_I Fd_J} | \mathcal{E} \rangle = -\frac{1}{2} \mathrm{trace} \left\{ T_{IJ} Q_{JJ}^{-1} T_{IJ}' E(\underline{s}_I \underline{s}_I') \right\} , \tag{4.14}$$

if any of the following conditions hold:

(a) $\underline{x}_I = 0$

(b) $T_{IM} = [0]$

(c) $E(\underline{x}_I) = \underline{0}$ and $T_{IJ} = [0]$.

Additivity Characteristics of the EVPIGPF

In the Entrepreneur's Decision of §3 the reader may observe that the value of simultaneous information on c and e does not equal the sum of the value of information on c and the value of information on e . Similary, the value of simultaneous flexibility on p and q does not equal the sum of the value of flexibility on p and the value of flexibility on q . Analysis of the quadratic decision problem allows us to explore the first order additivity characteristics of the EVPIGPF. For the following two corollaries we assume in addition that the conditional expectation of $\underline{s}$ is a linear function of the observable state variables.

COROLLARY 3: Suppose the random variables composing the vector $\underline{s}_I$ upon which clairvoyance is available may be partitioned into two vectors $\underline{s}_{I1}$ and $\underline{s}_{I2}$ that are independent. Then

$$\langle v_{Cs_I Fd_J} | \mathcal{E} \rangle = \langle v_{Cs_{I1} Fd_J} | \mathcal{E} \rangle + \langle v_{Cs_{I2} Fd_J} | \mathcal{E} \rangle . \tag{4.15}$$

PROOF: By assumption, $\underline{x} = E(\underline{s} | \underline{s}_I) = D\underline{s}_I$ for some matrix D . Denoting the covariance matrix of $\underline{s}_I$ by C_{II} , (4.11) becomes

$$\langle v_{Cs_I Fd_J} | \mathcal{E} \rangle = -\frac{1}{2} \mathrm{trace} \left\{ D' T_{NJ} Q_{JJ}^{-1} T_{NJ} D C_{II} \right\} . \tag{4.16}$$

For convenience we assume that the variables have been ordered so that

$$\underline{s} = \begin{bmatrix} \underline{s}_{I1} \\ \underline{s}_{I2} \\ \underline{s}_{\bar{I}} \end{bmatrix} . \tag{4.17}$$

The independence of $\underline{s}_{I1}$ and $\underline{s}_{I2}$ implies

$$C_{II} = \begin{bmatrix} C_{I1I1} & 0 \\ 0 & C_{I2I2} \end{bmatrix} . \tag{4.18}$$

The proof follows by algebraic substitution.

We say that decision vectors $\underline{d}_{J1}$ and $\underline{d}_{J2}$ do not interact if the value function may be expressed

$$v(\underline{s};\underline{d}_{J1},\underline{d}_{J2},\underline{d}_{\bar{J}}) = v_1(\underline{s};\underline{d}_{J1},\underline{d}_{\bar{J}}) + v_2(\underline{s};\underline{d}_{J2},\underline{d}_{\bar{J}}) . \tag{4.19}$$

<u>COROLLARY 4</u>: Suppose the decision variables composing the decision vector $\underline{d}_J$ for which flexibility is available may be partitioned into two vectors $\underline{d}_{J1}$ and $\underline{d}_{J2}$ which do not interact. Then

$$\langle v_{Cs_I Fd_J} | \mathcal{E} \rangle = \langle v_{Cs_I Fd_{J1}} | \mathcal{E} \rangle + \langle v_{Cs_I Fd_{J2}} | \mathcal{E} \rangle . \tag{4.20}$$

<u>PROOF</u>: For convenience we assume decision variables are ordered so that

$$\underline{d} = \begin{bmatrix} \underline{d}_{J1} \\ \underline{d}_{J2} \\ \underline{d}_{\bar{J}} \end{bmatrix} . \tag{4.21}$$

For the quadratic value function, the non-interaction assumption means that the matrix Q_{JJ} has the diagonal form

$$Q_{JJ} = \begin{bmatrix} Q_{J1J1} & 0 \\ 0 & Q_{J2J2} \end{bmatrix} . \tag{4.22}$$

Equation (4.20) follows by direct substitution into (4.16).

The results indicate that the first order additivity or non-additivity of the value of information is determined by state variable correlation. To a first order approximation, if two pieces of information are uncorrelated, then the value of obtaining that information simultaneously equals the sum of the values of receiving each item of information by itself. Similarly, the first order determinant of the additivity or non-additivity of the value of flexibility is decision variable interaction. If the value function is additive in two decision vectors, then, to first order, the value of simultaneously obtaining flexibility on both decision vectors will equal the sum of the values of obtaining flexibility on each vector individually.

Figures 4.1 and 4.2 illustrate these results for the special case of a four-variable quadratic decision with value function

$$v(s_1,s_2;d_1,d_2) = - d_1^2 - d_2^2 + 2qd_1d_2 + t_{11}s_1d_1 + t_{12}s_1d_2 + t_{21}s_2d_1 + t_{22}s_2d_2 \tag{4.23}$$

and normally distributed state variables. Figure 4.1 shows how the sign of $\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_2Fd_1d_2,\mathcal{E}\rangle$ depends on correlation ρ and interactions q . As we might expect, if correlation is high enough the sum of the values of individual information will exceed the value of joint information. Figure 4.2 shows the sign of $\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1s_2Fd_1,\mathcal{E}\rangle - \langle v|Cs_1s_2Fd_2,\mathcal{E}\rangle$ as a function of ρ and q . If decision variable interaction is high enough, we can expect the value of joint flexibility to exceed the sum of the values of individual flexibility.

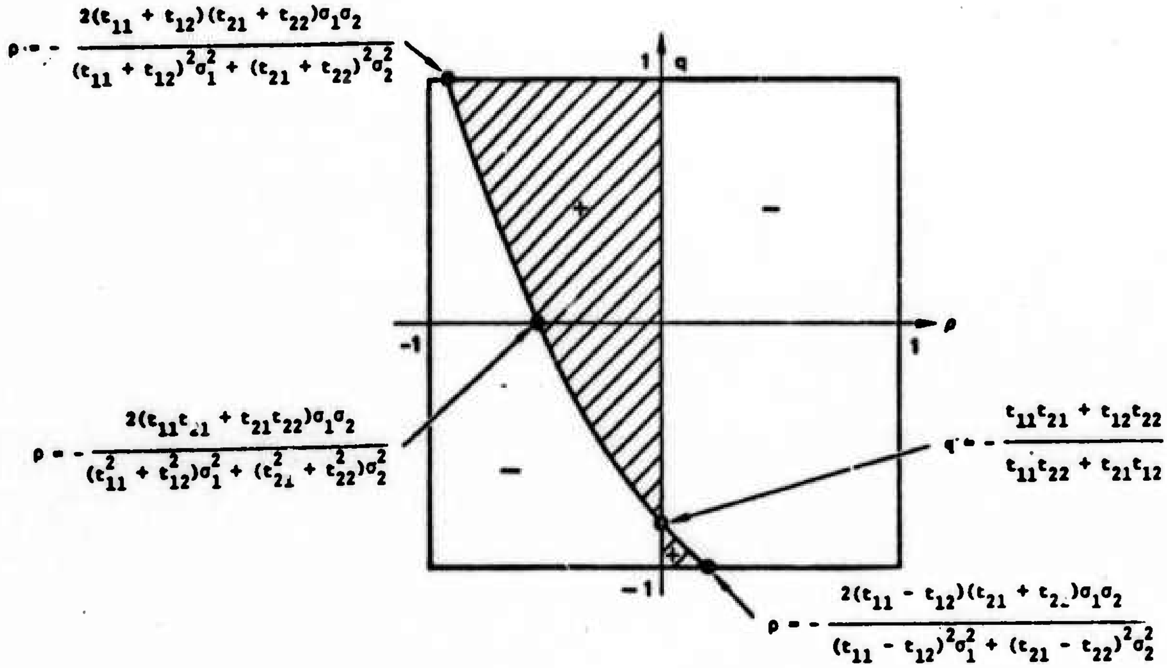

FIGURE 4.1 THE SIGN OF $\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_2Fd_1d_2,\mathcal{E}\rangle$ AS A FUNCTION OF ρ AND q FOR THE FOUR-VARIABLE PRODUCTION PROBLEM

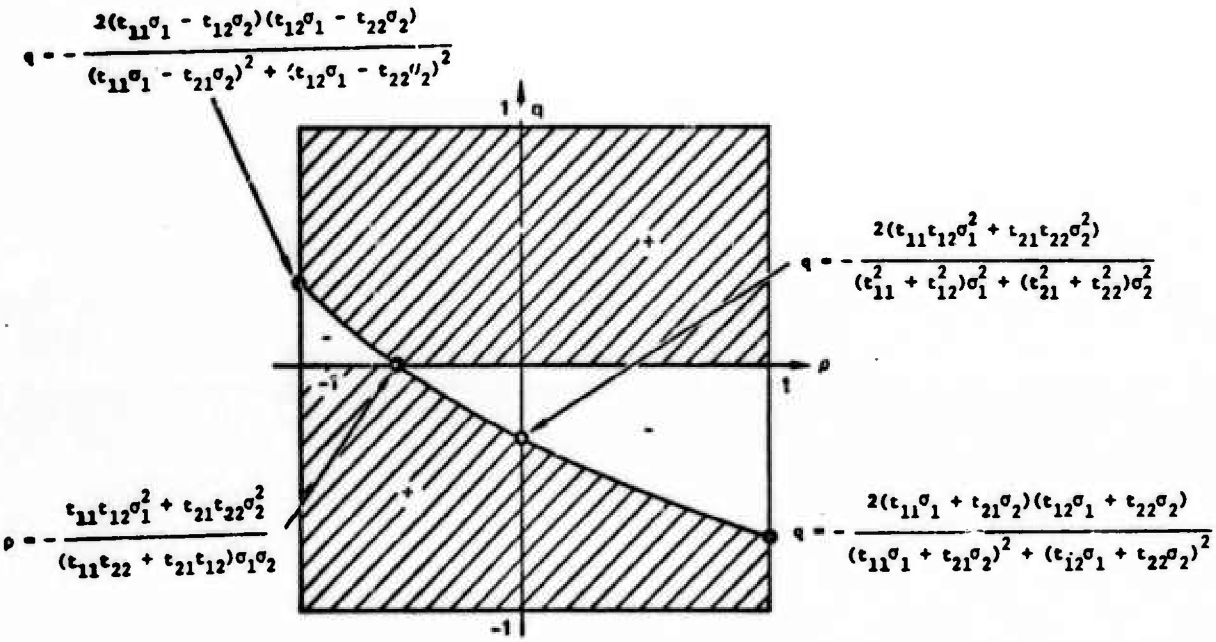

FIGURE 4.2 THE SIGN OF $\langle v|Cs_1s_2Fd_1d_2,\mathcal{E}\rangle - \langle v|Cs_1s_2Fd_1,\mathcal{E}\rangle - \langle v|Cs_1s_2Fd_2,\mathcal{E}\rangle$ AS A FUNCTION OF ρ AND q FOR THE FOUR-VARIABLE PRODUCTION PROBLEM

5. Proximal Analysis

The computational difficulty of performing the value of information given flexibility calculations gives impetus to a search for simplifying approximations. In this section we show that under certain conditions an approximation to the expected value of perfect information given perfect flexibility may be obtained by applying sensitivity analysis to the decision problem's deterministic value model. The technique is an extension to Howard's proximal decision model [3] .

In proximal decision analysis a quadratic equation in the first two moments of $\underline{s}$ is used to approximate the optimal decision strategy. Following Howard and Rice [6] we expand $v(\underline{s},\underline{d})$ in a second order Taylor series about the prior mean $\bar{\underline{s}}$ and the optimum deterministic decision $\hat{\underline{d}}$. We obtain Eq. (4.1) with

$$\underline{b} = \left[\frac{\partial v}{\partial s_i}\bigg|_{(\bar{\underline{s}},\hat{\underline{d}})}\right] , \tag{5.1}$$

$$W = \left[\frac{\partial^2 v}{\partial s_i \partial s_j}\bigg|_{(\bar{\underline{s}},\hat{\underline{d}})}\right] , \tag{5.2}$$

$$T = \left[\frac{\partial^2 v}{\partial s_i \partial d_j}\bigg|_{(\bar{\underline{s}},\hat{\underline{d}})}\right] , \tag{5.3}$$

$$Q = \left[\frac{\partial^2 v}{\partial d_i \partial d_j}\bigg|_{(\bar{\underline{s}},\hat{\underline{d}})}\right] . \tag{5.4}$$

$$\underline{r} = \left[\frac{\partial v}{\partial d_j}\bigg|_{(\bar{\underline{s}},\hat{\underline{d}})}\right] = 0 . \tag{5.5}$$

Open- and Partially Closed-Loop Sensitivities

Let the state and decision variable settings be incremented by amounts $\Delta\underline{s}$ and $\Delta\underline{d}$ from $\bar{\underline{s}}$ and $\hat{\underline{d}}$ respectively, then the approximate increase in v, denoted Δv, is given by

$$\Delta v = \underline{b}'\Delta\underline{s} + \frac{1}{2}\Delta\underline{s}'W\Delta\underline{s} + \Delta\underline{s}'T\Delta\underline{d} + \frac{1}{2}\Delta\underline{d}'Q\Delta\underline{d} \ . \tag{5.6}$$

We wish to find the open-loop sensitivity of v to changes in state variables s_i with i belonging to some index set I. The result is obtained from (5.6) with $\Delta\underline{d} = 0$, $\Delta s_k = 0$, $k \notin I$,

$$\Delta v_{ol} = \underline{b}_I'\Delta\underline{s}_I + \frac{1}{2}\Delta\underline{s}_I'W_{II}\Delta\underline{s}_I \ . \tag{5.7}$$

Next we calculate the partially closed-loop sensitivity in which the only decision variables that may be adjusted are those d_j with j in an index set J. Putting $\Delta\underline{s}_{\bar{I}}$ and $\Delta\underline{d}_{\bar{J}}$ equal to zero in (5.6) yields

$$\Delta v = \underline{b}_I'\Delta\underline{s}_I + \frac{1}{2}\Delta\underline{s}_I'W_{II}\Delta\underline{s}_I + \Delta\underline{s}_I'T_{IJ}\Delta\underline{d}_J + \frac{1}{2}\Delta\underline{d}_J'Q_{JJ}\Delta\underline{d}_J \ . \tag{5.8}$$

Setting the gradient with respect to $\underline{d}_J$ equal to zero, we get an expression showing how the flexible decision variables are optimally adjusted in response to changes in state variables:

$$\Delta\underline{d}_J^* = -Q_{JJ}^{-1}T_{IJ}'\Delta\underline{s}_I \ . \tag{5.9}$$

Substituting this expression into (5.8) gives the partially closed loop sensitivity of outcome value to state variable changes.

$$\Delta v_{cl} = \underbrace{\underline{b}_I'\Delta\underline{s}_I + \frac{1}{2}\Delta\underline{s}_I'W_{II}\Delta\underline{s}_I}_{\text{open-loop sensitivity}} \underbrace{- \frac{1}{2}\Delta\underline{s}_I'T_{IJ}Q_{JJ}^{-1}T_{IJ}'\Delta\underline{s}_I}_{\text{effect of compensation}} \ . \tag{5.10}$$

We see, in analogy with Howard's results [3, Equation 7.4], that the partially closed-loop sensitivity is composed of terms representing the open-loop sensitivity to state variables, plus terms that show the effect of compensation.

The Expected Value of Deterministic Compensation

Subtracting (5.7) from (5.10) and taking the expectation with respect to the marginal probability distribution of s_I, we obtain an expression for the expected value of deterministic compensation,

$$\langle v_{comp}|\mathcal{E}\rangle = -\frac{1}{2}E(\Delta\underline{s}_I' T_{IJ} Q_{JJ}^{-1} T_{IJ}' \Delta\underline{s}_I) = -\frac{1}{2}\mathrm{trace}[T_{IJ} Q_{JJ}^{-1} T_{IJ}' E(\Delta\underline{s}_I \Delta\underline{s}_I')] \; . \tag{5.11}$$

A comparison with (4.29) shows that (5.11) is exactly the expected value of perfect information on $\underline{s}_I$ given perfect flexibility on $\underline{d}_J$ for an expected value decision maker with a quadratic value function if any of the conditions of Corollary 2 are satisfied.

Now suppose that all state variables are adjusted in the sensitivity calculations. The compensation function becomes

$$v_{comp}(\Delta\underline{s}) = \Delta\underline{s}' T_{NJ} Q_{JJ}^{-1} T_{NJ}' \Delta\underline{s} \; . \tag{5.12}$$

If the function $E(\Delta\underline{s}|\Delta\underline{s}_I)$ is available, the compound function

$$v_{comp}[E(\Delta s|\Delta s_I)] = E(\Delta\underline{s}'|\Delta\underline{s}_I) T_{NJ} Q_{JJ}^{-1} T_{NJ}' E(\Delta\underline{s}|\underline{s}_I) \tag{5.13}$$

may be formed. Taking the expectation of (5.13) yields

$$\langle v_{comp}|\mathcal{E}\rangle = E[E(\Delta\underline{s}|\Delta\underline{s}_I) T_{NJ} Q_{JJ}^{-1} T_{NJ}' E(\Delta\underline{s}|\Delta\underline{s}_I)] \; , \tag{5.14}$$

which is the expected value of perfect information on $\underline{s}_I$ given perfect flexibility on $\underline{d}_J$ for an expected value decision maker with a quadratic value function if any of the conditions of Corollary 1 are satisfied.

Approximating the EVPIGPF with Sensitivity Analysis

Howard [3, Appendix B] gives a method for numerically evaluating $\underline{b}$, W, T, Q, and various conditional and unconditional covariance matrices for a complicated, many-variable, smooth value function. Hence, the proximal model and the theorem and corollaries of § 4 provide a means for obtaining an approximation to the expected value of information given flexibility.

The above results, however, show that under certain conditions a simpler procedure may be applied. For the purpose of illustration, assume that the value function for the decision model contains two state variables and two decision variables. We wish to estimate the value of perfect information on s_1 given perfect flexibility on d_2. For the first calculation we shall ignore the effect that knowledge of s_1 has on the estimation of s_2. The procedure consists of:

1. evaluating deterministic open-loop sensitivity to changes in the observable state variable s_1,
2. evaluating deterministic partially closed-loop sensitivity (d_2 continuously optimized) to changes in s_1,
3. calculating the difference in these two functions, $v_{comp}(\Delta s_1)$,
4. determining the expectation of v_{comp}.

If knowledge of s_1 impacts the decision through its effect on the estimation of s_2, this may be included in the approximation using the following procedure:

1. evaluate deterministic open-loop joint sensitivity to changes in s_1 and s_2,

2 evaluate deterministic partially closed-loop joint sensitivity (d_2 continuously optimized) to changes in s_1 and s_2 ,

3. calculate the difference in these two functions, $v_{comp}(\Delta s_1, \Delta s_2)$,

4. determine $E(\Delta s_2 | \Delta s_1)$, the conditional mean of Δs_2 as a function of Δs_1 ,

5. determine the expected value of $v_{comp}[\Delta s_1, E(\Delta s_2 | \Delta s_1)]$.

Implementation of this procedure could be facilitated by approximating joint sensitivities with quadratic functions. A good approximation may be expected provided that $E[E(\Delta s_2 | \Delta s_1)] = 0$; that is, the prior expectation is a zero shift in the mean of the unobservable state variable.

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